Journal of Engineering Physics and Thermophysics, Vol. 81, No. 5, 2008

EFFECTS OF FREE CONVECTION ON THE OSCILLATORY FLOW OF A POLAR FLUID THROUGH A POROUS MEDIUM IN THE PRESENCE OF VARIABLE WALL HEAT FLUX

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UDC 536.25

The purpose of this work is to study a laminar two-dimensional free convective oscillatory flow of an incompressible polar fluid through a saturated porous medium occupying a semi-infinite region of the space which is bounded by an infinite vertical permeable plate in the presence of oscillating suction and variable wall heat flux. The governing equations are based on the local volume averaging technique. The generalized Darcy equation accounting for polar effects is employed. Analytical expressions for the linear momentum, angular momentum, and energy fields are obtained by using the two-term perturbation technique. The numerically computed results are shown graphically and compared with the corresponding ones for a viscous (Newtonian) fluid. Distribution of the mean velocity of a polar fluid is found to increase as compared to the Newtonian fluid. The analysis reveals multiple boundary-layer structure for velocity.

Introduction. The buoyancy-induced flows in fluid-saturated porous media have been a prime topic of many studies during the past three and a half decades. This is now considered to be an important field in the general areas of fluid mechanics and heat transfer in view of the importance in various engineering and technological applications, such as heat transfer associated with storage of nuclear waste, exothermic reaction in packed-bed reactors, heat removal from nuclear fuel debris, underground coal gasification, ground water hydrology, heat recovery from geothermal systems, large storage systems of agricultural products, etc.

A comprehensive review of literature regarding the subject mentioned was reported in recent books by Ingham and Pop [1], Nield and Bejan [2], Vafai [3], Pop and Ingham [4], and Ingham et al. [5]. The study of boundary-layer phenomena is of great importance in recent times owing to their wide applications in several engineering fields. The boundary-layer zone can be considered to be an interface region where fluid flow and heat transfer characteristics of two different porous media and a fluid or of porous and impermeable media are adjusted to one another. To give a specific example, one can consider flow from petroleum reservoirs, wherein the oil flow encounters different layers of sand, rock, shale, lime stone, etc. Vafai and Thiyagaraja [6] analyzed the flow and heat transfer at the interface region of porous medium. The analysis of natural convection about a vertical plate embedded in a porous medium was examined by Kim and Vafai [7]. Vafai and Kim [8] obtained an exact solution for the interface region between a porous medium and a fluid layer.

Studies on the response of a laminar boundary-layer flow due to free-stream oscillations are of prime importance in many industrial and aerodynamic flow problems. Typical problems arise in the study of aircraft response to atmospheric gusts, aerofoil lift hysteresis at the stall, flutter phenomena involving wing, panel, and stalling flutter, as well as the prediction of flow over helicopter rotor blades and through turbo-machinery blade cascades. Lighthill [9] initiated the study of such flows dealing with the response of a boundary layer to external unsteady fluctuations about a mean value. Stuart [10] investigated the features of interest in the case of an oscillatory flow over an infinite plate with constant suction. Messiha [11] examined the laminar boundary-layer flow in the presence of fluctuating suction in phase with fluctuations of a main stream over a non-zero mean. Kafousias et al. [12, 13] analyzed the free convection effects with and without MHD effects on the oscillatory flow in the Stokes problem past an infinite porous vertical limiting surface with constant suction. The influence on the oscillatory flow in the presence of free convective flow through a porous medium was studied by Raptis and Perdikis [14]. Gholizadeh [15, 16] investigated the effects of

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1062-0125/08/8105-0905

MHD oscillatory flow past a vertical porous plate through porous medium in the presence of thermal and mass diffusion with constant heat source. Rudraiah et al. [17] considered the effect of oscillatory mixed convection in a vertical porous stratum. Goma and Taweel [18] examined the effects of oscillatory motion on heat transfer at vertical surfaces and developed a model that predicted both transient and time average heat-transfer rates. Effects of unsteady mixed convection boundary-layer flow along a symmetric wedge with variable surface temperature were investigated by Hossain et al. [19]. In all these studies only viscous (Newtonian) fluids were considered.

Recently, interest in problems of non-Newtonian fluids has grown considerably because of many applications in chemical process industries, food preservation techniques, petroleum production, and in power engineering. Many industrial fluids are non-Newtonian in nature and their characteristics are considered to be rheological. Particularly, slurries (china clay and coal in water, sewage sludge, etc.), multiphase mixtures (oil-water emulsions, gas-liquid dispersions such as froths and foams, butter) are non-Newtonian fluids. Further examples displaying a variety of non-Newtonian characteristics include pharmaceutical formulations, cosmetics and toiletries, paints, synthetic lubricants, biological fluids (blood, synovial fluid saliva), and foodstuffs (jams, jellies, soaps, marmalades). The non-Newtonian boundary-layer flows through a porous medium with heat transfer are encountered in such wide applications as fluid film lubrication, analysis of polymer in chemical engineering, and fossil fuels which may saturate underground beds displaying non-Newtonian behavior. Kaloni [20] examined the fluctuating flow of an elastico-viscous fluid past a porous flat plate. Soundalgekar and Puri [21, 22] investigated the laminar velocity and thermal boundary layers in fluctuating flow of an elastico-viscous fluid past an infinite plate with variable suction. Oscillatory free convective flow of such fluid past an impulsively started infinite vertical porous plate was examined by Singh [23]. Pascal [24] studied the rheological effects of non-Newtonian behavior of displacing fluids on stability of a moving interface in radial oil displacement mechanism in porous media. Nakayama and Koyama [25] examined the buoyancy-induced flow of non-Newtonian fluids over a nonisothermal body of arbitrary shape in a fluid-saturated porous medium. Mehta and Rao [26, 27] studied the buoyancy-induced flow of non-Newtonian fluids over a nonisothermal horizontal/vertical plate embedded in a porous medium. The effect of thermal development of forced convection in a channel occupied by a porous medium which is saturated by a non-Newtonian fluid was examined by Nield and Kuznetsov [28]. Hang Xu et al. [29] obtained the series solution for unsteady boundary-layer flows of non-Newtonian fluids near the forward stagnation point. Odell and Haward [30] reported the results on viscosity enhancement in non-Newtonian flow of dilute aqueous polymer solutions through crystal and random porous media.

The fluids which sustain couple stresses, so-called polar fluids, are the fluids with microstructures which are mechanically significant, when the characteristic dimension of the problem is of the same order of magnitude as the size of the microstructure. Extensive reviews of the corresponding theory can be found in the review article by Cowin [31]. Since the micro-structure size is the same as the average pore size, it is relevant to study the flow of polar fluids through a porous medium. Examples of fluids which can be modeled as polar are slurries, polymer solutions, mud, crude oil, body fluids, lubricants with polymer additives, etc. The effects of couple stresses on the flow through a porous medium were investigated by Raptis [32] and Patil and Hiremath [33]. Hiremath and Patil [34] analyzed the free convection effects on the oscillatory flow of couple stress fluids through a porous medium. In this analysis, the effect of couple stresses in the Darcy resistance was not considered. Sharma and Gupta [35] examined the effects of thermal convection in micropolar fluids in porous medium. Effect of rotation on thermal convection in micropolar fluids through a porous medium was studied by Sharma and Kumar [36]. Raptis and Takhar [37] examined steady flow of a polar fluid through a porous medium using the generalized Forchheimer's model. Sharma and Thakur [38] analyzed the effects of MHD on couple stress fluid heated from below in porous medium. V. Sharma and S. Sharma [39] discussed effects of the thermosolutal convection of micropolar fluids with MHD through a porous medium. Kim [40] examined the unsteady convection flow effects of micropolar fluids through a porous medium, where free-stream flow consisted of a mean velocity and superimposed exponentially small variation with time. Kim [41] analyzed the unsteady MHD convection flow of polar fluids past a vertical moving porous plate in a porous medium. Ibrahim and Hassanein [42] obtained local nonsimilarity solutions for mixed convection boundary-layer flow of a micropolar fluid on horizontal flat plate with variable surface temperature. Linear and nonlinear analyses of convection in a micropolar fluid occupying a high-porosity medium were studied by Siddheshwar and Krishna [43]. Ibrahim et al. [44] examined the effects of unsteady MHD micropolar fluid flow over a vertical porous plate through a porous medium in the presence of thermal and mass diffusion with a constant heat source. The effect of heat and mass transfer on MHD micro-

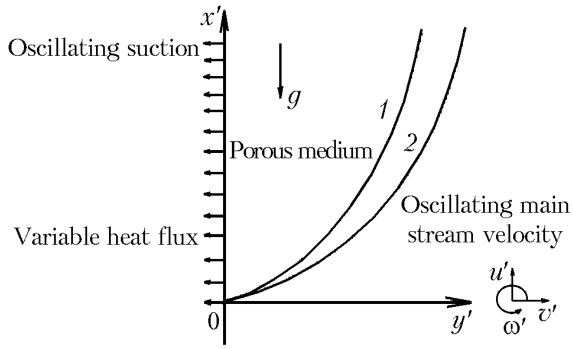

Fig. 1. Schematic diagram and coordinate system for hydrodynamic (1) and temperature (2) boundary layers.

polar flow over a vertical moving porous plate in a porous medium was studied by Kim [45]. Hassanein et al. [46] examined the effects of natural convection flow of micropolar fluid from a permeable surface with uniform heat flux in porous medium. R. C. Sharma and M. Sharma [47] discussed the flow of the couple stress fluid which contained suspended particles heated and soluted from below in porous medium. Ogulu [48] studied the effect of oscillating temperature flow of a polar fluid past a vertical porous plate in the presence of couple stresses and radiation. Rehman and Sattar [49] analyzed the effects of magnetohydrodynamic convective flow of a micropolar fluid past a continuously moving porous plate in the presence of heat generation/absorption. The effect of rotation on a layer of micropolar ferromagnetic fluid which was heated from below saturating a porous medium was investigated by Sunil et al. [50].

In the present analysis, the effect of variable wall heat flux on the oscillatory free convective flow of a polar fluid through a porous medium with oscillating suction at the wall is studied. Examples of the physical situation presented are: (i) heat removal of nuclear fuel debris buried in the deep sea-bed and (ii) heat recovery from geothermal systems. The flow takes place near a hot vertical plate bounding the porous region which is filled with water containing soluble and insoluble chemical materials. Such a fluid is modeled as a polar fluid. The flow is due to the buoyancy forces generated by the temperature gradient. The analysis is carried out under the assumption of a constant porosity of the medium, i.e., neglecting the effect of possible porosity variations near the wall (which is often discussed for beds of packed spheres [51]). The generalized Darcy equation accounting for effects of couple stresses is employed.

Mathematical Formulation. We consider the two-dimensional oscillatory free convective flow of a laminar, incompressible, polar fluid through a porous medium occupying a semi-infinite region of the space bounded by an infinite vertical permeable plate with oscillating suction and variable heat flux applied at the wall. Far away from the plate, the free-stream velocity $U'_\infty(t)$ oscillates over a constant non-zero mean value U_∞ in the direction of the plate. The X' axis is taken along the plate and Y' axis is normal to the plate. The geometry of the flow problem is shown in Fig. 1. Let $(u', v', 0)$ and $(0, 0, \omega')$ denote velocity and angular velocity fields, respectively. It is assumed that the viscous and Darcy resistance terms are taken into account with constant permeability of a porous medium, that is to say, the variable fluid properties are negligible except for the buoyancy term which is directly responsible for the fluid motion. Due to the assumption of infinite plate, the flow variables, except pressure p, are functions of y' and t' only. The governing equations, for the boundary-layer flow, are given by Aero et al. [52] and D'ep [53] as equations of continuity, linear momentum, angular momentum, and energy, respectively:

$$\frac{\partial v'}{\partial y'} = 0 , \tag{1}$$

$$\frac{\partial u'}{\partial t'} + v' \frac{\partial u'}{\partial y'} = -\frac{1}{\rho}\frac{\partial p}{\partial x'} + (\nu + \nu_r)\frac{\partial^2 u'}{\partial y'^2} + 2\nu_r \frac{\partial \omega'}{\partial y'} - \frac{\nu + \nu_r}{K'} u' + g\beta_0 (T' - T'_\infty) , \tag{2}$$

$$\frac{\partial \omega'}{\partial t'} + v' \frac{\partial \omega'}{\partial y'} = \frac{\gamma}{I} \frac{\partial^2 \omega'}{\partial y'^2} , \tag{3}$$

$$\frac{\partial T'}{\partial t'} + v' \frac{\partial T'}{\partial y'} = \frac{\lambda}{\rho C_p} \frac{\partial^2 T'}{\partial y'^2} , \tag{4}$$

where $\gamma = (C_a + C_d)/I$. The fourth term on the RHS of Eq. (2) is the modified Darcy term, which accounts for the polar effects due to the couple stresses experienced by the fluid; it is obtained by the volume averaging technique defined in [54]. Further, in the energy equation (4) the terms describing the heat due to viscous and the couple stress dissipation as well as Darcy dissipation are disregarded (a first approximation).

Following D'ep [53], we write the relevant boundary conditions:

$$y' = 0 : \;\; u' = 0 \, , \;\; \frac{\partial \omega'}{\partial y'} = -\frac{\partial^2 u'}{\partial y'^2} \, , \;\; \frac{\partial T'}{\partial y'} = -\frac{q'}{\lambda}\left(1 + \varepsilon B \exp\,(in't')\right) ;$$

$$y' \to \infty : \;\; u' \to U_\infty \left(1 + \varepsilon \exp\,(in't')\right) , \;\; \omega' \to 0 \, , \;\; T' \to T'_\infty \, , \tag{5}$$

where B is a real positive constant and ε is a small parameter such that $\varepsilon B \le 1$ and $i = \sqrt{-1}$. These are derived from the assumption that the couple stresses are dominant during the rotation of the particles.

For the fluctuating suction at the wall, we assume

$$v' = -v_0 \left(1 + \varepsilon A \exp\,(in't')\right) \tag{6}$$

where v_0 is a non-zero constant mean suction velocity and A is a real positive constant such that $\varepsilon A \le 1$. Only the real part of the physical quantities involved in the mathematical analysis has physical meaning. The negative sign in Eq. (6) indicates that the suction velocity is directed towards the plate. The continuity equation (1) yields the solution

$$v'(y', t') = -v_0 \left(1 + \varepsilon A \exp\,(in't')\right) , \tag{6a}$$

which satisfies condition (6) at the wall.

In view of Eq. (6a), Eqs. (2)–(4) become

$$\frac{\partial u'}{\partial t'} - v_0 \left(1 + \varepsilon A \exp\,(in't')\right) \frac{\partial u'}{\partial y'} = -\frac{1}{\rho}\frac{\partial p}{\partial x'} + (\nu + \nu_r)\frac{\partial^2 u'}{\partial y'^2} + 2\nu_r \frac{\partial \omega'}{\partial y'} - \frac{\nu + \nu_r}{K'} u' + g\beta_0 (T' - T'_\infty) \, , \tag{7}$$

$$\frac{\partial \omega'}{\partial t'} - v_0 \left(1 + \varepsilon A \exp\,(in't')\right) \frac{\partial \omega'}{\partial y'} = \frac{\gamma}{I} \frac{\partial^2 \omega'}{\partial y'^2} , \tag{8}$$

$$\frac{\partial T'}{\partial t'} - v_0 \left(1 + \varepsilon A \exp\,(in't')\right) \frac{\partial T'}{\partial y'} = \frac{\lambda}{\rho C_p} \frac{\partial^2 T'}{\partial y'^2} . \tag{9}$$

For the free stream, the momentum equation (2) takes the form

$$\frac{dU'_\infty}{dt'} = -\frac{1}{\rho'}\frac{\partial p}{\partial x'} - \frac{\nu + \nu_r}{K'} U'_\infty . \tag{10}$$

Eliminating the pressure gradient between Eqs. (7) and (10), we have

$$\frac{\partial u'}{\partial t'} - v_0\left(1 + \varepsilon A \exp(in't')\right)\frac{\partial u'}{\partial y'} = \frac{dU'_\infty}{dt'} + (\nu + \nu_r)\frac{\partial^2 u'}{\partial y'^2} + 2\nu_r \frac{\partial \omega'}{\partial y'} + \frac{\nu + \nu_r}{K'}(U'_\infty - u') + \beta_0 g (T' - T'_\infty)\,. \tag{11}$$

Now we introduce nondimensional variables as follows:

$$y = \frac{y' v_0}{\nu}\,,\quad t = \frac{v_0^2 t'}{\nu}\,,\quad n = \frac{\nu n'}{v_0^2}\,,\quad u = \frac{u'}{U_\infty}\,,\quad U = \frac{U'_\infty}{U_\infty}\,,\quad \omega = \frac{\nu \omega'}{U_\infty v_0}\,,\quad \alpha = \frac{\nu_r}{\nu}\,,\quad \beta = \frac{I\nu}{\gamma}\,,$$

$$K = \frac{v_0^2 K'}{\nu^2}\,,\quad T = \frac{(T' - T'_\infty)\,\lambda v_0}{(T'_w - T'_\infty)\, q' \nu}\,,\quad \Pr = \frac{\rho \nu C_p}{\lambda}\,,\quad \mathrm{Gr} = \frac{\nu g \beta_0 (T'_w - T'_\infty)}{U_\infty v_0^2}\,. \tag{12}$$

With the help of Eq. (12), the governing equations (11), (8), and (9) are reduced to the following nondimensional form:

$$\frac{\partial u}{\partial t} - \left(1 + \varepsilon A \exp(int)\right)\frac{\partial u}{\partial y} = \frac{dU}{dt} + (1 + \alpha)\frac{\partial^2 u}{\partial y^2} + 2\alpha \frac{\partial \omega}{\partial y} + \frac{1 + \alpha}{K}(U - u) + \mathrm{Gr}\, T\,, \tag{13}$$

$$\frac{\partial \omega}{\partial t} - \left(1 + \varepsilon A \exp(int)\right)\frac{\partial \omega}{\partial y} = \frac{1}{\beta}\frac{\partial^2 \omega}{\partial y^2}\,, \tag{14}$$

$$\frac{\partial T}{\partial t} - \left(1 + \varepsilon A \exp(int)\right)\frac{\partial T}{\partial y} = \frac{1}{\Pr}\frac{\partial^2 T}{\partial y^2}\,. \tag{15}$$

The relevant boundary conditions (5) reduce to

$$y = 0:\quad u = 0\,,\quad \frac{\partial \omega}{\partial y} = -\frac{\partial^2 u}{\partial y^2}\,,\quad \frac{\partial T}{\partial y} = -\left(1 + \varepsilon B \exp(int)\right);$$

$$y \to \infty:\quad u \to U\,,\quad \omega \to 0\,,\quad T \to 0\,. \tag{16}$$

For fluctuations in the free-stream velocity U'_∞, we set

$$U'_\infty(t) = U_\infty\left(1 + \varepsilon \exp(in't')\right), \tag{17}$$

which after nondimensionalisation becomes

$$U = 1 + \varepsilon \exp(int)\,. \tag{18}$$

Equations (6) and (17) imply that the main stream velocity fluctuations are assumed to be in phase with the suction velocityfluctuations.

Solution of the Problem. In order to reduce the system of partial differential equations (13)–(15) to a system of ordinary differential equations, we assume that the fields of linear and angular velocities and temperature are given as

$$u = u_0(y) + \varepsilon \exp(int)\, u_1(y) + o(\varepsilon^2) + \ldots\,, \tag{19}$$

$$\omega = \omega_0(y) + \varepsilon \exp(int)\, \omega_1(y) + o(\varepsilon^2) + \dots , \tag{20}$$

$$T = T_0(y) + \varepsilon \exp(int)\, T_1(y) + o(\varepsilon^2) + \dots . \tag{21}$$

Substituting Eqs. (19)–(21) into Eqs. (13)–(15) respectively, equating the harmonic and nonharmonic terms, and disregarding the coefficients of the order of $o(\varepsilon^2)$ we obtain the system of equations for the velocity, angular velocity, and temperature fields.

Steady basic flow. This flow is described by equations

$$(1+\alpha)\, u_0'' + u_0' - \frac{1+\alpha}{K} u_0 = -\left\{ 2\alpha\omega_0' + \mathrm{Gr}\, T_0 + \frac{1+\alpha}{K} \right\} , \tag{22}$$

$$\omega_0'' + \beta\omega_0' = 0 , \tag{23}$$

$$T_0'' + \mathrm{Pr}\, T_0' = 0 , \tag{24}$$

subjected to the corresponding reduced boundary conditions

$$\begin{aligned} & y = 0 : \;\; u_0 = 0 , \;\; \omega_0' = -\, u_0'' , \;\; T_0' = -\,1 ; \\ & y \to \infty : \;\; u_0 \to 1 , \;\; \omega_0 \to 0 , \;\; T_0 \to 0 . \end{aligned} \tag{25}$$

Unsteady oscillatory flow. Here,

$$(1+\alpha)\, u_1'' + u_1' - \left(\frac{1+\alpha}{K} + in \right) u_1 = -\left\{ Au_0' + 2\alpha\omega_1' + \mathrm{Gr}\, T_1 + \left(\frac{1+\alpha}{K} + in \right) \right\} , \tag{26}$$

$$\omega_1'' + \beta\omega_1' - \beta in\omega_1 = -\, \beta A \omega_0' , \tag{27}$$

$$T_1'' + \mathrm{Pr}\, T_1' - \mathrm{Pr}\, inT_1 = -\, \mathrm{Pr}\, AT_0' , \tag{28}$$

and the corresponding reduced boundary conditions are

$$\begin{aligned} & y = 0 : \;\; u_1 = 0 , \;\; \omega_1' = -\, u_1'' , \;\; T_1' = -\, B ; \\ & y \to \infty : \;\; u_1 \to 1 , \;\; \omega_1 \to 0 , \;\; T_1 \to 0 , \end{aligned} \tag{29}$$

where primes denote differentiation with respect to y.

The solutions of the differential equations (22)–(24) and (26)–(28) with the boundary conditions (25) and (29), respectively, are given by the expressions

$$u_0(y) = 1 + C_2 \exp(R_1 y) + A_1 C_1 \exp(-\,\beta y) + A_2 \exp(-\,\mathrm{Pr}\, y) , \tag{30}$$

$$\omega_0(y) = C_1 \exp(-\,\beta y) , \tag{31}$$

$$T_0(y) = \exp(-\Pr y)/\Pr , \tag{32}$$

$$u_1(y) = 1 + C_4 \exp(R_7 y) + C_3 A_4 \exp(R_5 y) + A_5 \exp(R_3 y) + A_6 \exp(R_1 y) + + A_7 \exp(-\beta y) + A_8 \exp(-\Pr y) , \tag{33}$$

$$\omega_1(y) = C_3 \exp(R_5 y) + A_3 \exp(-\beta y) , \tag{34}$$

$$T_1(y) = D_1 \exp(R_3 y) + D_2 \exp(-\Pr y) , \tag{35}$$

where

$$R_{1,2} = \frac{-1 \mp \sqrt{1 + \frac{(1+\alpha)^2}{K}}}{2(1+\alpha)} , \quad R_{3,4} = \frac{-\Pr \mp \sqrt{\Pr^2 + 4in\Pr}}{2} , \quad R_{5,6} = \frac{-\beta \mp \sqrt{\beta^2 + 4in\beta}}{2} ,$$

$$R_{7,8} = \frac{-1 \mp \left[1 + 4(1+\alpha)\left(in + \frac{1+\alpha}{K}\right)\right]^{1/2}}{2(1+\alpha)} , \quad A_1 = \frac{2\alpha\beta}{(1+\alpha)(\beta + R_1)(\beta + R_2)} ,$$

$$A_2 = -\frac{\mathrm{Gr}}{\Pr(1+\alpha)(\Pr + R_1)(\Pr + R_2)} , \quad A_3 = \frac{iA\beta C_1}{n} , \quad A_4 = -\frac{2\alpha R_5}{(1+\alpha)(R_5 - R_7)(R_5 - R_8)} ,$$

$$A_5 = -\frac{\mathrm{Gr}\, D_1}{(1+\alpha)(R_3 - R_7)(R_3 - R_8)} , \quad A_6 = -\frac{AR_1 C_2}{(1+\alpha)(R_1 - R_7)(R_1 - R_8)} ,$$

$$A_7 = \frac{2\alpha\beta A_3 + A\beta A_1 C_1}{(1+\alpha)(\beta + R_7)(\beta + R_8)} , \quad A_8 = \frac{A \Pr A_2 - \mathrm{Gr}\, D_2}{(1+\alpha)(\Pr + R_7)(\Pr + R_8)} ,$$

$$C_1 = \frac{(1 + A_2) R_1^2 - A_2 \Pr^2}{(\beta^2 - R_1^2) A_1 - \beta} , \quad C_2 = \frac{A_1 A_2 \Pr^2 - (1 + A_2)(\beta^2 A_1 - \beta)}{(\beta^2 - R_1^2) A_1 - \beta} ,$$

$$C_3 = \frac{H_2 - H_1 R_7^2}{A_4 (R_5^2 - R_7^2) + R_5} , \quad C_4 = \frac{H_1 (A_4 R_5^2 + R_5) - A_4 H_2}{A_4 (R_5^2 - R_7^2) + R_5} , \quad D_1 = (\Pr D_2 - B) R_3^{-1} , \quad D_2 = \frac{iA}{n} ,$$

$$H_1 = -(1 + A_5 + A_6 + A_7 + A_8) ,$$

$$H_2 = -(R_3^2 A_5 + R_1^2 A_6 + \beta^2 A_7 + \Pr^2 A_8 - \beta A_3) .$$

Let the quantities u_1 and ω_1, which can be presented as

$$u_1 = u_r + iu_i\,, \quad \omega_1 = \omega_r + i\omega_i, \tag{36}$$

as well as $|u_1|$ and $|\omega_1|$, defined as

$$|u_1| = \left(u_r^2 + u_i^2\right)^{1/2}, \quad |\omega_1| = \left(\omega_r^2 + \omega_i^2\right)^{1/2}, \tag{37}$$

be proportional to the magnitudes of velocity fluctuations and angular velocity fluctuations, respectively. The expressions

$$\Phi_u = \arctan \frac{u_i}{u_r}\,, \quad \Phi_\omega = \arctan \frac{\omega_i}{\omega_r} \tag{38}$$

represent the phase angles of velocity and angular velocity fluctuations with respect to suction and free-stream velocity fluctuations. The mathematical expressions for u_r, u_i, ω_r, and ω_i are presented in the Appendix.

Coefficient of Skin Friction. The primary physical quantity of interest is the skin friction at the wall which is often of great importance in technological applications. With the help of the velocity field in the boundary layer, the skin friction at the wall of the plate is given by

$$\tau_w = \frac{\tau'_w}{\rho U_\infty v_0} = \left(\frac{\partial u}{\partial y}\right)_{y=0} = \left(\frac{du_0}{dy} + \varepsilon \exp(int)\frac{du_1}{dy}\right)_{y=0} = \left(\frac{du_0}{dy}\right)_{y=0} + \varepsilon\,|C_f|\,\cos(nt + \Phi_C)\,, \tag{39}$$

where $|C_f| = (C_r^2 + C_i^2)^{1/2}$ is the amplitude of skin friction oscillations and $\varphi_C = \arctan(C_i/C_r)$ is the phase angle. The mathematical expressions for u'_0, C_r, and C_i are also presented in the Appendix.

Results and Discussion. For physical insight into the flow problem considered herein, the numerical computation of the distributions of velocity u and angular velocity ω is carried out for different values of the material parameters α and β, permeability parameter K, Grashof number Gr, Prandtl number Pr, wall suction parameter A, wall heat flux parameter B, and frequency n. The Newtonian case corresponds to $\alpha = 0$ and $\beta = 0$. The numerically computed results for the distributions of the mean velocity u_0 and the mean angular velocity ω_0 are plotted in Figs. 2–5. From Fig. 2 we observe that both u_0 and ω_0 (in magnitude) increase with α. The negative values of the angular velocity ω_0 indicate that the microrotation of the substructures in the polar fluid is clockwise. The mean velocity is found to increase in the case of polar fluid as compared to that for the Newtonian one (curve 4 in Fig. 2a). The effect of β on the velocities can be readily seen from Fig. 3: as β increases, they decrease. Figure 4 illustrates the effect of Grashof number Gr on the distributions of velocities. The values of Gr have been chosen so that they could be of interest from the physical point of view. Natural convection sets in due to the temperature difference $(T'_w - T'_\infty)$. When $(T'_w - T'_\infty) > 0$ (i.e., Gr > 0) and $(T'_w - T'_\infty) < 0$ (i.e., Gr < 0)*, the situation corresponds to cooling and heating of the permeable plate by free convection currents, respectively. The numerical results obtained for Gr = 0 correspond to the case of the absence of free convection currents. It is observed from Fig. 4 that an increase in the cooling of the plate (Gr > 0) leads to a rise in the mean velocity and angular velocity (in magnitude), while the velocities fall, when the heating of the plate increases (Gr < 0). In Fig. 5, the effect of permeability K of the porous medium is presented. It is seen that the velocities decrease with increasing K.

The influence of the Prandtl number Pr on the velocities is also investigated. It is shown that the increasing value of Pr leads to a decrease in the velocities. The numerical results also show that an increase in Pr results in the reduction of the boundary layer thickness. The reason is that smaller values of Pr are equivalent to an increase in thermal conductivity and therefore heat can diffuse away from the heated plate more quickly than in the case of greater values of Pr. Hence, the boundary layer is thicker and the rate of heat transfer is lower for higher Pr.

The distributions of the magnitudes $|u_1|$ and $|\omega_1|$ are shown in Figs. 6—8. Figure 6 illustrates the distributions of $|u_1|$ and $|\omega_1|$ for different values of the suction parameter A. It is observed that $|u_1|$ and $|\omega_1|$ increase with A near the wall and slowly decrease farther away (but for $|u_1|$ the influence of A is insignificant). The effects of the heat flux

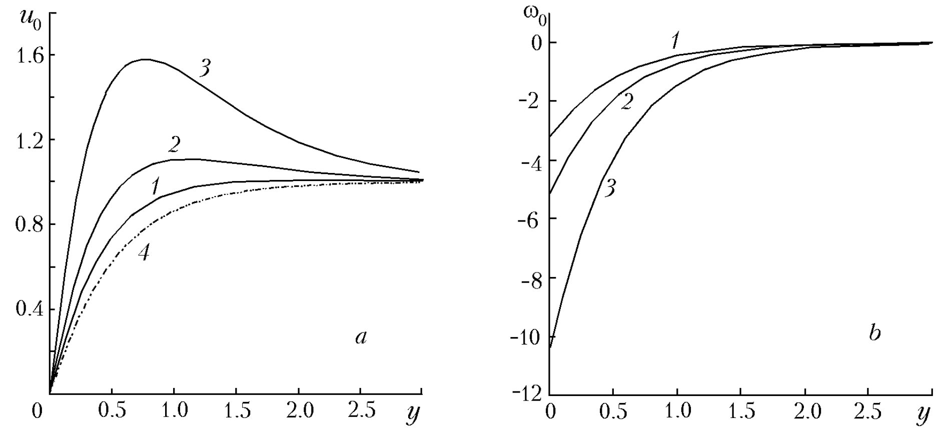

Fig. 2. Profiles of mean (a) and angular (b) velocities for different values of the material parameter α at β = 2, Gr =2, K = 0.5, Pr = 2: 1) α = 0.1; 2) 0.3; 3) 0.5; 4) 0.

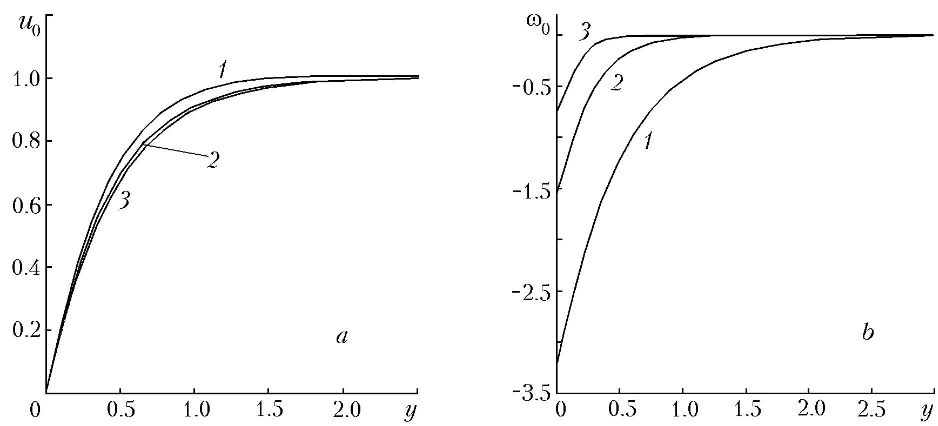

Fig. 3. Profiles of mean (a) and angular (b) velocities for different values of the material parameter β at α = 0.1, Gr =2, K = 0.5, Pr = 2: 1) β = 2; 2) 4; 3) 8.

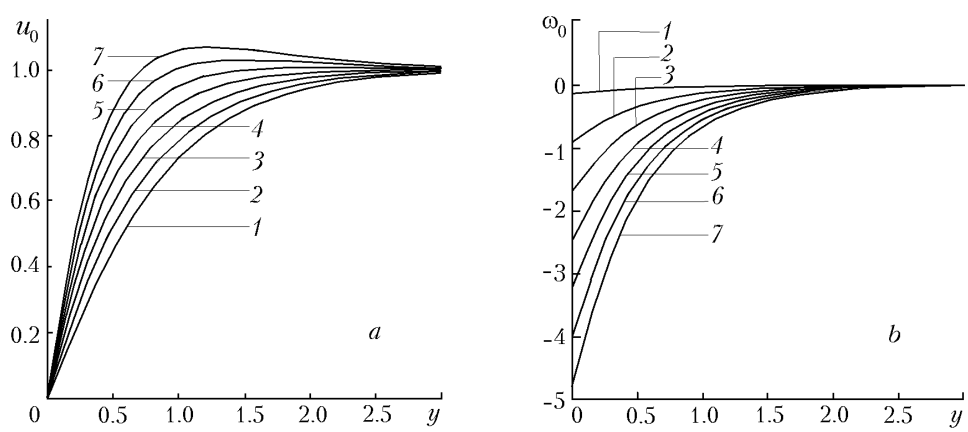

Fig. 4. Profiles of mean (a) and angular (b) velocities for different values of Gr at α = 0.1, β = 2, K = 0.5, Pr = 2: 1) Gr = –6; 2) –4; 3) –2; 4) 0; 5) 2; 6) 4; 7) 6.

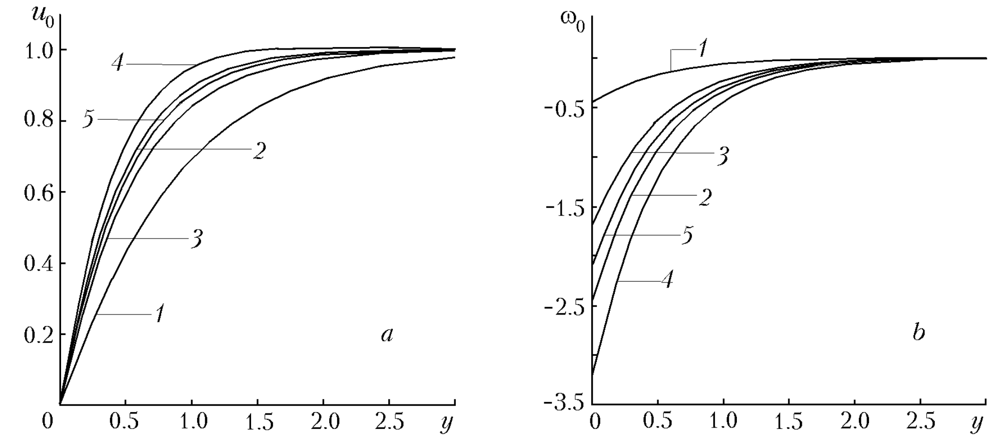

Fig. 5. Profiles of mean (a) and angular (b) velocities for different values of Gr and K at $\alpha = 0.1$, $\beta = 2$, Pr = 2: 1) Gr = 0 and $K = 0.5$; 2) –2 and 0.5; 3) –2 and 1.5; 4) 2 and 0.5; 5) 2 and 1.5.

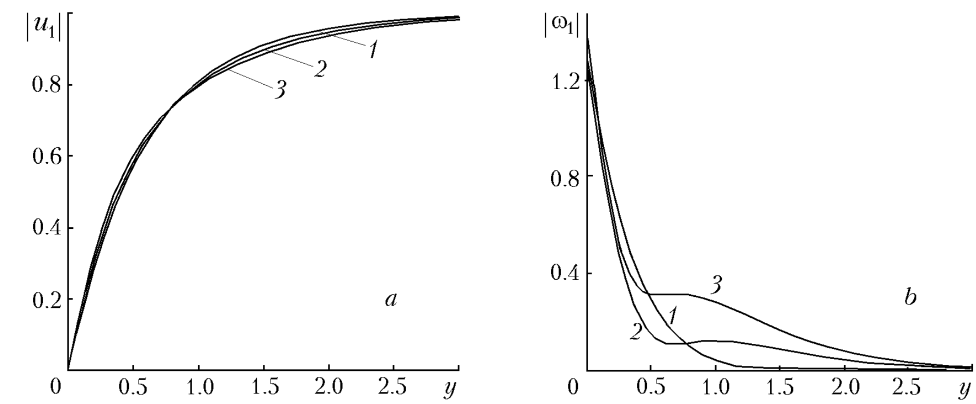

Fig. 6. Profiles of magnitudes of fluctuations of mean (a) and angular (b) velocities for different values of the suction parameter A at $\alpha = 0.1$, = 2, Gr =2, $K = 0.5$, Pr = 2, $B = 0.1$, $n = 1$: 1) $A = 0.2$; 2) 0.5; 3) 0.8.

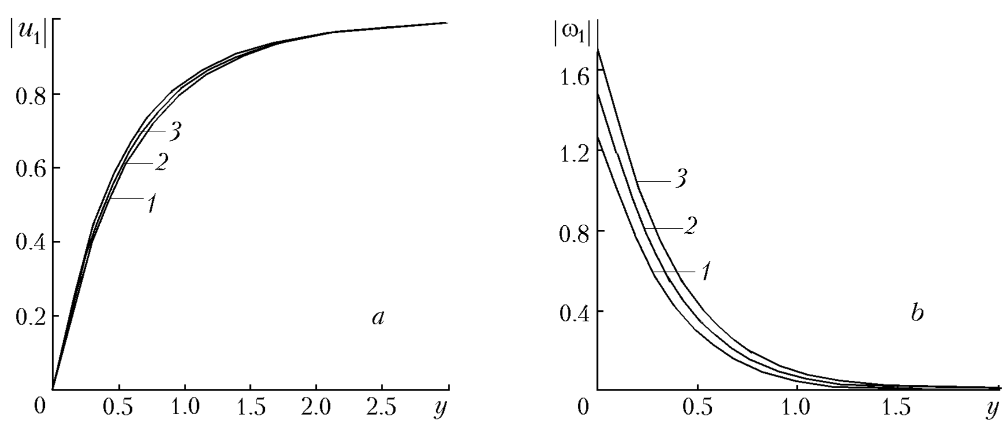

Fig. 7. Profiles of magnitudes of fluctuations of mean (a) and angular (b) velocities for different values of the heat flux parameter B at $\alpha = 0.1$, $\beta = 2$, Gr =2, $K = 0.5$, Pr = 2, $A = 0.2$, $n = 1$: 1) $B = 0.1$; 2) 0.5; 3) 0.9.

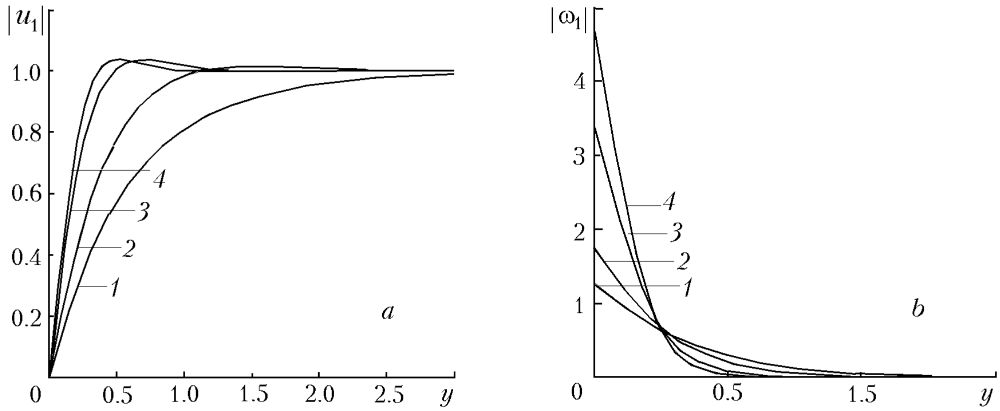

Fig. 8. Profiles of magnitudes of fluctuations of mean (a) and angular (b) velocities for different values of frequency at $\alpha = 0.1$, $\beta = 2$, Gr = 2, $K = 0.5$, Pr = 2, $A = 0.2$, $B = 0.1$: 1) $n = 1$; 2) 10; 3) 50; 4)100.

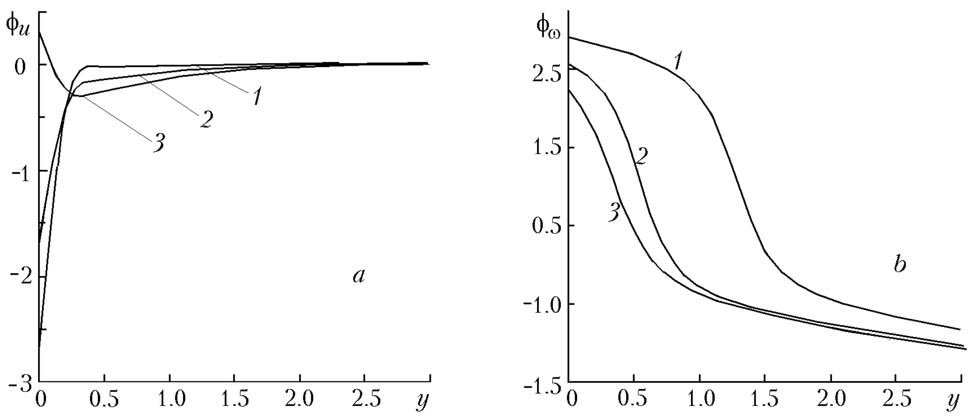

Fig. 9. Profiles of phase angle of fluctuations of mean (a) and angular (b) velocities for different values of the suction parameter A at $\alpha = 0.1$, $\beta = 2$, Gr = 2, $K = 0.5$, Pr = 2, $B = 0.1$, $n = 1$: 1) $A = 0.2$; 2) 0.5; 3) 0.8. The curves correspond to the same values of A as in Fig. 6.

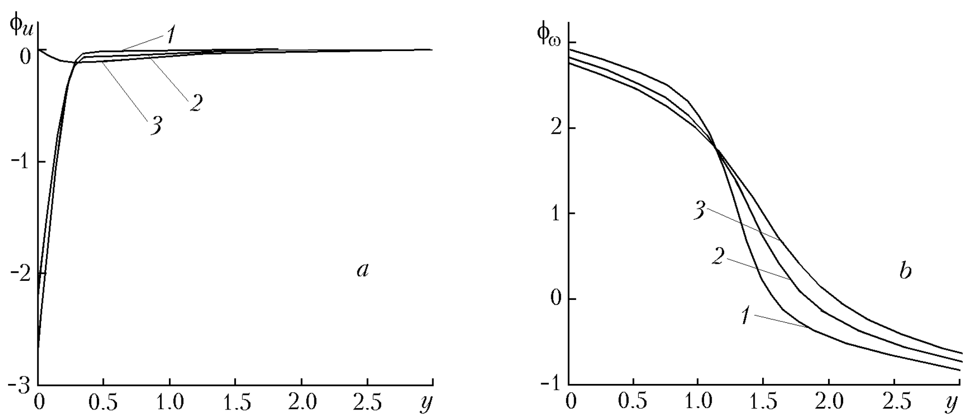

Fig. 10. Profiles of phase angle of fluctuations of mean (a) and angular (b) velocities for different values of the heat flux parameter B at $\alpha = 0.1$, $\beta = 2$, Gr = 2, $K = 0.5$, Pr = 2, $A = 0.2$, $n = 1$: 1) $B = 0.1$; 2) 0.5; 3) 0.9. The curves correspond to the same values of B as in Fig. 7.

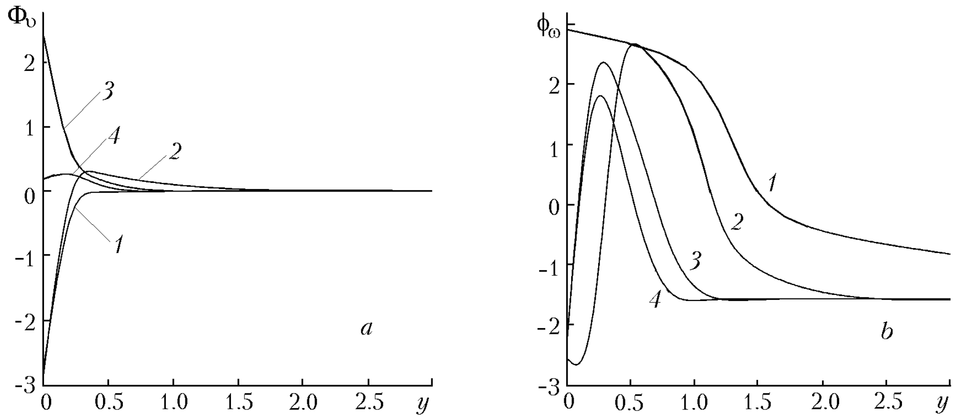

Fig. 11. Profiles of phase angle of fluctuations of mean (a) and angular (b) velocities for different values of frequency at α = 0.1, β = 2, Gr =2, K = 0.5, Pr = 2, A = 0.2, B = 0.1: 1) n = 1; 2) 10; 3) 50; 4)100. The curves correspond to the same values of α as in Fig. 8.

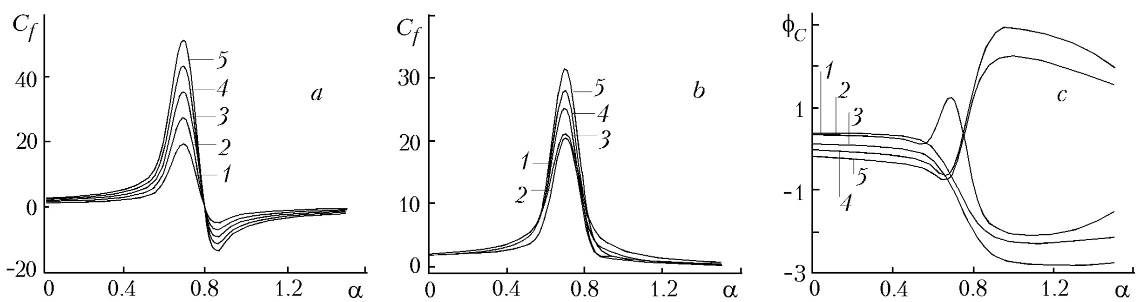

Fig. 12. Dependence of the mean skin friction (a), its magnitude (b), and phase angle (c) on the material parameter α for different values of Gr at β = 2, K = 0.5, Pr = 2, A = 0.2, B = 0.1, n = 1: 1) Gr = –4; 2) –2; 3) 0; 4) 2; 5) 4.

parameter B can be readily seen in Fig. 7: $|u_1|$ and $|\omega_1|$ increase with B (and here too $|u_1|$ changes moderately). This indicates that the flow behavior can be enhanced by applying a wall heat flux, which may be of use in the recovery of hydrocarbons from underground petroleum deposits. Heat is injected into a reservoir in the form of a hot water or steam or heat can be generated by burning a part of the crude in the reservoir. In all such thermal recovery processes, fluid flow through the porous medium takes place and the convection currents are detrimental. Figure 8 shows the profiles of $|u_1|$ and $|\omega_1|$ for different frequencies n. It is seen that $|u_1|$ and $|\omega_1|$ increase with n near the wall and decrease as the flow goes away from the wall. The relative influences of the parameters α, β, Gr, Pr, and K on $|u_1|$ and $|\omega_1|$ are also investigated but not presented. It follows from the numerical results that $|u_1|$ and $|\omega_1|$ increase with α and Pr, while they decrease as β, K, and Gr increase. Further, it is observed that there is a profound influence of the Prandtl number on the flow near the wall.

The profiles of phase angles of fluctuations of velocity ϕ_u and angular velocity ϕ_ω (in radians) are depicted in Figs. 9—11. Figure 9 gives ϕ_u and ϕ_ω for different values of the wall suction parameter A. The angle ϕ_u has a phase lead ($\phi_u > 0$) at the wall for large values of A, whereas ϕ_ω is always characterized by a phase lead at the wall and a phase lag ($\phi_u < 0$) away from this. In Fig. 10, we observe that ϕ_u and ϕ_ω decrease as B increases. Here, ϕ_u has a phase lag near the wall for larger B, while ϕ_ω is characterized by a phase lead. The effect of frequency n on ϕ_u and ϕ_ω is seen in Fig. 11. For larger frequencies, the phase angles ϕ_u and ϕ_ω have phase lead and phase lag, respectively. As seen from the numerical results, ϕ_u and ϕ_ω decrease as α increases, while they increase with β, K, Gr, and Pr.

The numerical results for the skin friction are presented in Fig. 12. The variation of the mean skin friction on the permeable plate with the material parameter α for different values of Grashof number is given in Fig. 12a. This figure shows an anomalous behavior at critical values of α: when $\alpha \approx 0.6$, C_f abruptly increases, drastically falls at $\alpha \approx 0.8$, and then becomes parallel to the axis. As the Grashof number increases, this anomalous behavior becomes more significant. The cooling of the plate increases and the heating decreases for an increasing value of Gr. The magnitude of skin friction $|C_f|$ can be seen from Fig. 12b. Here, a similar anomalous behavior is evident. The dependences of the phase angle of skin friction ϕ_c on α are presented in Fig. 12c. The figure shows that the dependences mentioned are opposite in nature for cooling (Gr > 0) and heating (Gr < 0) of the plate.

The velocity and angular velocity fields exhibit multiple boundary-layer structures. In the basic steady part of the flow (see Eq. (30)) there is the boundary layer of thicknesses of the orders of $|R_1|^{-1}$, β^{-1}, and Pr^{-1}. The angular velocity and temperature fields are confined to the sublayer of thicknesses of the order of β^{-1} (Eq. (31)) and Pr^{-1} (Eq. (32)), respectively. In the unsteady oscillatory part of the flow (Eq. (33)), Stokes sublayers of thicknesses of orders $|R_{3r}|^{-1}$, $|R_{5r}|^{-1}$, and $|R_{7r}|^{-1}$ exist, where the subscript r relates to the real part which essentially is due to the fluctuations in the wall suction and free-stream velocity. In the angular velocity field there are sublayers of thicknesses of the orders of $|R_{5r}|^{-1}$ and β^{-1} (Eq. (34)), whereas for the temperature field the thicknesses of the orders of $|R_{3r}|^{-1}$ and Pr^{-1} take place (Eq. (35)).

Conclusions. We have examined the governing equations for an oscillatory free convective flow of a polar fluid through a porous medium in the presence of oscillating suction and variable heat flux applied at the wall. It is shown that the flow characteristics are profoundly influenced by the material parameters α and β of the polar fluid, permeability K of the porous medium, Grashof number Gr which characterizes the free convection currents, Prandtl number Pr, wall suction parameter A, wall heat flux parameter B, and the frequency n. The results obtained herein may be useful in petroleum engineering, especially in the techniques of oil-recovery enhancement by application of wall suction and wall heat flux in the presence of disturbances in the free-stream flow. The suction and heat flux at the wall can be accounted for in the design of production plants of crude oil and geothermal power.

NOTATION

A, suction velocity parameter; B, heat flux parameter; C_a, C_d, coefficients of couple stress viscosities; C_f, skin friction coefficient; C_p, specific heat at constant pressure; Gr, Grashof number; g, acceleration due to gravity; I, constant of dimensions of the moment of inertia of unit mass; K' and K, permeability and dimensionless permeability of the porous medium; n' and n, frequency and dimensionless frequency of oscillations; Pr, Prandtl number; p, pressure; T', temperature and dimensionless temperature in the boundary layer; T'_∞, temperature of the fluid far away from the plate; T'_w, temperature at the wall; t' and t, time and dimensionless time; U_∞, mean free-stream velocity; U'_∞, free-stream velocity; u' and v', components of velocities along and perpendicular to the plate, respectively; u, dimensionless velocity; u_0, mean velocity; u_1, fluctuating part of the velocity; v_0, mean suction velocity; x' and y', coordinates along and perpendicular to the plate; y, dimensionless coordinate perpendicular to the plate; α, β, material parameters characterizing the polarity of the fluid; β_0, coefficient of volumetric expansion of the fluid; ε, perturbation parameter; λ, thermal conductivity of the fluid; ν, kinematic viscosity of the fluid; ν_r, rotational kinematic viscosity of the fluid; ρ, density of the fluid; ρ_∞, density of the fluid far away from the surface; τ'_w and τ', skin friction and dimensionless skin friction at the wall; ϕ_u, ϕ_ω, and ϕ_c, phase angles of fluctuations of velocity, angular velocity, and skin friction; ω' and ω, angular and dimensionless angular velocity; ω_0, mean angular velocity; ω_1, fluctuating part of the angular velocity. Subscripts: f, friction; r and i, real and imaginary parts; w, wall; 0 and 1, steady basic and unsteady oscillatory flows.

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Appendix:

The mathematical expressions appeared in Eqs. (36), (37), and (39) are as follows:

$$u_r = 1 + \exp(R_{7r}y)\,(C_{4r}\cos(R_{7i}y) - C_{4i}\sin(R_{7i}y)$$

$$+ \exp(R_{5r}y)\,(S_{12}\cos(R_{5i}y) - T_{12}\sin(R_{5i}y)) + \exp(R_{3r}y)\,(A_{5r}\cos(R_{3i}y) - A_{5i}\sin(R_{3i}y))$$

$$+ A_{6r}\exp(R_1 y) + A_{7r}\exp(-\beta y) + A_{8r}\exp(-\Pr y)\,,$$

$$u_i = \exp(R_{7r}y)\,(C_{4i}\cos(R_{7i}y) + C_{4r}\sin(R_{7i}y))$$

$$+ \exp(R_{5r}y)\,(T_{12}\cos(R_{5i}y) + S_{12}\sin(R_{5i}y)) + \exp(R_{3r}y)\,(A_{5i}\cos(R_{3i}y) + A_{5r}\sin(R_{3i}y))$$

$$+ A_{6i}\exp(R_1 y) + A_{7i}\exp(-\beta y) + A_{8i}\exp(-\Pr y)\,,$$

$$w_r = \exp(R_{5r}y)\,(C_{3r}\cos(R_{5i}y) - C_{3i}\sin(R_{5i}y))\,,$$

$$w_i = \exp(R_{5r}y)\,(C_{3i}\cos(R_{5i}y) + C_{3r}\sin(R_{5i}y)) + A_{3i}\exp(-\beta y)\,,$$

$$M_{3,4} = \left[\frac{1}{2}\left(\pm \Pr^2 + \sqrt{\Pr^4 + 16\Pr^2 n^2}\right)\right]^{1/2},$$

$$R_{3r} = -\frac{\Pr + M_3}{2}\,,\quad R_{3i} = -\frac{M_4}{2}\,,\quad R_{4r} = \frac{M_3 - \Pr}{2}\,,\quad R_{4i} = -R_{3i}\,,$$

$$M_{5,6} = \left[\frac{1}{2}\left(\pm \beta^2 + \sqrt{\beta^4 + 16\beta^2 n^2}\right)\right]^{1/2},$$

$$R_{5r} = -\frac{\beta + M_5}{2}\,,\quad R_{5i} = -\frac{M_6}{2}\,,\quad R_{6r} = \frac{M_5 - \beta}{2}\,,\quad R_{6i} = -R_{5i}\,,$$

$$M_{7,8} = \left[\frac{1}{2}\left(\pm\left(1 + \frac{4\,(1+\alpha)^2}{K}\right) + \sqrt{(1 + 4/K\,(1+\alpha)^2)^2 + 16n^2\,(1+\alpha)^2}\right)\right]^{1/2},$$

$$R_{7r} = -\frac{1 + M_7}{2}\,,\quad R_{7i} = -\frac{M_8}{2}\,,\quad R_{8r} = \frac{M_7 - 1}{2}\,,\quad R_{8i} = -R_{7i}\,,$$

$$D_{1r} = \frac{R_{3r}S_1 + R_{3i}T_1}{R_{3r}^2 + R_{3i}^2}\,,\quad D_{1i} = \frac{R_{3r}T_1 - R_{3i}S_1}{R_{3r}^2 + R_{3i}^2}\,,\quad D_{2r} = 0\,,\quad D_{2i} = \frac{A}{n}\,,$$

$$S_1 = \Pr D_{2r} - B\,,\quad T_1 = \Pr D_{2i}\,,\quad A_{3r} = 0\,,\quad A_{3i} = A\beta C_1 n^{-1}\,,$$

$$S_2 = (1+\alpha)\,((R_{5r} - R_{7r})\,(R_{5r} - R_{8r}) - (R_{5i} - R_{7i})\,(R_{5i} - R_{8i}))\,,$$

$$T_2 = (1+\alpha)\,((R_{5i} - R_{7i})\,(R_{5r} - R_{8r}) + (R_{5r} - R_{7r})\,(R_{5i} - R_{8i}))\,,$$

$$A_{4r} = -\frac{2\alpha\,(R_{5r}S_2 + R_{5i}T_2)}{S_2^2 + T_2^2}\,, \quad A_{4i} = -\frac{2\alpha\,(R_{5i}S_2 - R_{5r}T_2)}{S_2^2 + T_2^2}\,,$$

$$S_3 = (1+\alpha)\,((R_{3r} - R_{7r})\,(R_{3r} - R_{8r}) - (R_{3i} - R_{7i})\,(R_{3i} - R_{8i}))\,,$$

$$T_3 = (1+\alpha)\,((R_{3i} - R_{7i})\,(R_{3r} - R_{8r}) + (R_{3r} - R_{7r})\,(R_{3i} - R_{8i}))\,,$$

$$A_{5r} = -\frac{\mathrm{Gr}\,(D_{1r}S_3 + D_{1i}T_3)}{S_3^2 + T_3^2}\,, \quad A_{5i} = -\frac{\mathrm{Gr}\,(D_{1i}S_3 - D_{1r}T_3)}{S_3^2 + T_3^2}\,,$$

$$S_4 = (1+\alpha)\,((R_1 - R_{7r})\,(R_1 - R_{8r}) - R_{7i}R_{8i})\,, \quad T_4 = (1+\alpha)\,((R_1 - R_{8r})\,R_{7i} + (R_1 - R_{7r})\,R_{8i})\,,$$

$$A_{6r} = -\frac{AR_1C_2S_4}{S_4^2 + T_4^2}\,, \quad A_{6i} = \frac{AR_1C_2T_4}{S_4^2 + T_4^2}\,,$$

$$S_5 = (1+\alpha)\,((\beta + R_{7r})\,(\beta + R_{8r}) - R_{7i}R_{8i})\,, \quad T_5 = (1+\alpha)\,((\beta + R_{8r})\,R_{7i} + (\beta + R_{7r})\,R_{8i})\,,$$

$$S_6 = A\beta A_1C_1S_5 + 2\alpha\beta A_{3i}T_5\,, \quad T_6 = 2\alpha\beta A_{3i}S_5 - A\beta A_1C_1T_5\,,$$

$$S_7 = (1+\alpha)\,((\mathrm{Pr} + R_{7r})\,(\mathrm{Pr} + R_{8r}) - R_{7i}R_{8i})\,, \quad A_{7r} = \frac{S_6}{S_5^2 + T_5^2}\,, \quad A_{7i} = \frac{T_6}{S_5^2 + T_5^2}\,,$$

$$T_7 = (1+\alpha)\,((\mathrm{Pr} + R_{8r})\,R_{7i} + (\mathrm{Pr} + R_{7r})\,R_{8i})\,,$$

$$S_8 = A\,\mathrm{Pr}\,A_2S_7 - \mathrm{Gr}\,D_{1i}T_7\,, \quad T_8 = -\,(\mathrm{Gr}\,D_{1i}S_7 + A\,\mathrm{Pr}\,A_2T_7)\,,$$

$$A_{8r} = \frac{S_8}{S_7^2 + T_7^2}\,, \quad A_{8i} = \frac{T_8}{S_7^2 + T_7^2}\,,$$

$$H_{4r} = -\,(1 + A_{5r} + A_{6r} + A_{7r} + A_{8r})\,, \quad H_{4i} = -\,(A_{5i} + A_{6i} + A_{7i} + A_{8i})\,,$$

$$S_9 = \left(R_{3r}^2 - R_{3i}^2\right)A_{5r} - 2R_{3r}R_{3i}A_{5i}\,, \quad T_9 = \left(R_{3r}^2 - R_{3i}^2\right)A_{5i} + 2R_{3r}R_{3i}A_{5r}\,,$$

$$H_{5r} = -\left(S_9 + R_1^2A_{6r} + \beta^2A_{7r} + \mathrm{Pr}^2A_{8r} - \beta A_{3r}\right),$$

$$H_{5i} = -\left(T_9 + R_1^2 A_{6i} + \beta^2 A_{7i} + \mathrm{Pr}^2 A_{8i} - \beta A_{3i}\right),$$

$$H_{6r} = H_{5r} - \left(H_{4r}\left(R_{7r}^2 - R_{7i}^2\right) - 2H_{4i}R_{7r}R_{7i}\right),$$

$$H_{6i} = H_{5i} - \left(H_{4i}\left(R_{7r}^2 - R_{7i}^2\right) + 2H_{4r}R_{7r}R_{7i}\right),$$

$$S_{10} = R_{5r}^2 - R_{5i}^2 - R_{7r}^2 + R_{7i}^2\,, \quad T_{10} = 2\,(R_{5r}R_{5i} - R_{7r}R_{7i})\,,$$

$$H_{7r} = R_{5r} + S_{10}A_{4r} - T_{10}A_{4i}\,, \quad H_{7i} = R_{5i} + T_{10}A_{4r} + S_{10}A_{4i}\,,$$

$$S_{11} = R_{5r} + \left(R_{5r}^2 - R_{5i}^2\right)A_{4r} - 2R_{5r}R_{5i}A_{4i}\,, \quad T_{11} = R_{5i} + \left(R_{5r}^2 - R_{5i}^2\right)A_{4i} + 2R_{5r}R_{5i}A_{4r}\,,$$

$$H_{8r} = H_{4r}S_{11} - H_{4i}T_{11} - A_{4r}H_{5r} + A_{4i}H_{5i}\,, \quad H_{8i} = H_{4i}S_{11} + H_{4r}T_{11} - A_{4r}H_{5i} - A_{4i}H_{5r}\,,$$

$$C_{3r} = \frac{H_{6r}H_{7r} + H_{6i}H_{7i}}{H_{7r}^2 + H_{7i}^2}\,, \quad C_{3i} = \frac{H_{6i}H_{7r} - H_{6r}H_{7i}}{H_{7r}^2 + H_{7i}^2}\,, \quad C_{4r} = \frac{H_{8r}H_{7r} + H_{8i}H_{7i}}{H_{7r}^2 + H_{7i}^2}\,,$$

$$C_{4i} = \frac{H_{8i}H_{7r} - H_{8r}H_{7i}}{H_{7r}^2 + H_{7i}^2}\,, \quad S_{12} = A_{4r}C_{3r} - A_{4i}C_{3i}\,, \quad T_{12} = A_{4i}C_{3r} + A_{4r}C_{3i}\,,$$

$$u_0' = R_1C_2 - \beta A_1C_1 - \mathrm{Pr}\,A_2\,, \quad S_{13} = R_{5r}A_{4r} - R_{5i}A_{4i}\,, \quad T_{13} = R_{5i}A_{4r} + R_{5r}A_{4i}\,,$$

$$C_r = R_{7r}C_{4r} - R_{7i}C_{4i} + S_{13}C_{3r} - C_{3i}T_{13} + R_{3r}A_{5r} - R_{3i}A_{5i} + R_1A_{6r} - \beta A_{7r} - \mathrm{Pr}\,A_{8r}\,,$$

$$C_i = R_{7i}C_{4r} + R_{7r}C_{4i} + T_{13}C_{3r} + C_{3i}S_{13} + R_{3i}A_{5r} + R_{3r}A_{5i} + R_1A_{6i} - \beta A_{7i} - \mathrm{Pr}\,A_{8i}\,.$$